Published by Institute of Physics Publishing for SISSA
Received: *July 25, 2007*
Accepted: *September 11, 2007*
Published: *September 24, 2007*

Reduced sigma-model on $O(N)$: Hamiltonian analysis and Poisson bracket of Lax connection

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Abstract: This short note is devoted to the study of the Hamiltonian formalism and the integrability of the bosonic model introduced in [hep-th/0612079]. We calculate Poisson bracket of spatial components of Lax connection and we argue that its structure implies classical integrability of the theory.

Keywords: AdS-CFT Correspondence, Bosonic Strings.

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http://jhep.sissa.it/archive/papers/jhep092007100/jhep092007100.pdf

Contents

1. Introduction and summary

The sigma model on $AdS_5 \times S^5$ [1] is complicated interacting theory whose solution is currently beyond the reach.[1] On the other hand recently J. Maldacena and I. Swanson proposed a relatively simple kinematical truncation of this theory [12]. Technical simplifications allow us to test some conjectures considering integrability of the string theory on $AdS_5 \times S^5$.[2] More precisely, the sigma model in this limit leads (after gauge fixing) to simpler toy model that is not, however, Lorentz-invariant theory in $1+1$ dimensions. On the other hand it is well defined system on its own. In fact, this system was carefully analysed in [12] where world-sheet S-matrix was also discussed. Moreover, it was demonstrated an classical integrability of the $O(N)$ sigma model in the near flat space limit in the sense that the Lax pair was constructed. This is very interesting result since now we have Lax pair for completely gauge fixed theory where the Virasoro constraints were solved. Since the form of the Lax pair is rather unusual it is interesting to study the property of this theory further.

It is well known that the existence of Lax connection implies the existence of an infinite tower of conserved charges in the classical theory. However, as was stressed recently in [24] this does not quite coincide with the standard definition of integrability. Integrability in the standard sense requires not only the existence of a tower of conserved charges but also requires that these charges are in "involution". In other words, these conserved charges Poisson brackets commute with each other.

However there is a long-standing problem in determining the Poisson brackets of the conserved charged for classical string theory formulated on background that admits Lax connection. Namely, the problem is due to the presence of *Non-Ultra Local* terms in the Poisson brackets of the world-sheet fields that lead to ambiguities in brackets for the

[1] For alternative pure-spinor approach description of superstring theory on $AdS_5 \times S^5$, see [2 – 11].

[2] For some related works, see [13 – 20].

charges.[3] As was shown recently in [24] a resolution of this problem is based on earlier work of Maillet [22, 23] where he proposed the regularisation the problematic brackets. Then this procedure was applied to the simplest classical subsector of the $AdS_5 \times S^5$ geometry in [24]. It was shown that this prescription leads to a very natural symplectic structure on the space of finite-gap solutions of the string equations of motion that were constructed in [27]. Then it was shown in [28] that the string theory on $AdS_5 \times S^5$ possesses infinite number of conserved charges that are in involution even on the world-sheet with general metric. Explicitly, the Poisson bracket of spatial components of Lax connection was calculated and it was shown that its form does not depend on world-sheet metric and takes precisely the form as in [27]. However it is important to stress that this analysis was valid in case when either the diffeomorphism invariance of theory was preserved or the metric components were fixed in some general form while the symmetry generated by Virasoro constraints was preserved. On the other hand it is not clear whether integrability is preserved in case of completely fixed theory, as for example, string theory in uniform light-cone gauge [29 − 31].[4] As the first step in the answering of this question we would like to calculate the Poisson bracket of the spatial components of Lax connection for simpler model introduced in [12]. It turns out that even if the resulting Poisson bracket is very complicated one can map it, following [22, 23] to the form that shows that the theory possesses an infinite number of local charges that are in involutions. We mean that this is a nice result that shows that the integrability persists in case of complete fixed theory as well.

We can extend this work in various ways. For example, it would be nice to perform the same analysis for the supersymmetric form of the model given in [12]. Then we would like to calculate Poisson brackets of Lax connection for bosonic string in uniform light-cone gauge.

The organisation of this paper is as follows. In next section 2 we review the procedure presented in [12] for $O(N)$ sigma model that leads to simpler model with completely fixed symmetry. In section 3 we review the construction of Lax connection for this system. Then in section 4 we present the Hamiltonian formalism of this system and we calculate the Poisson brackets of spatial components of Lax connection. Using these results we argue for integrability of the theory. Finally, in appendix (A) we review Maillet's treatment of the monodromy matrix and Lax connection.

2. Reduced $O(N)$ Sigma Model

In this section we review the analysis presented in [12] that leads to interesting new $1+1$ dimensional field theory. We consider $O(N)$ sigma model. The target space of this sigma model is a sphere S^{N-1}. Let us consider a state with a constant spin density $J = J_{12}$, where J_{kl} are rotation generators in the kl plane. Then we begin with the action

$$
\begin{aligned}
S &= -\frac{\sqrt{\lambda}}{4\pi} \int d\tilde{\sigma}^0 d\tilde{\sigma}^1 \sqrt{-\eta}(-\eta^{\alpha\beta}\partial_\alpha t \partial_\beta t + \eta^{\alpha\beta}\partial_\alpha x^m \partial_\beta x^n) \,, \\
x^m x^n \delta_{mn} &= 1 \,, \quad m = 1, \dots, N \,,
\end{aligned}
\tag{2.1}
$$

[3]For recent discussion of these problems in the context of string theory on $AdS_5 \times S^5$, see [15, 25, 26].

[4]This gauge was also discussed in [32, 33].

where $\partial_\alpha \equiv \frac{\partial}{\partial \tilde{\sigma}^\alpha}$, $\alpha = 0, 1$ and we work in conformal gauge with the world-sheet metric $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1)$. To have a contact with [12] we introduce the parameter

$$g = \frac{\sqrt{\lambda}}{4\pi} \ . \tag{2.2}$$

For simplicity, we start with the case when $N = 3$ and parametrize S^2 as

$$x^1 = \sin\theta \, , \quad x^2 = \cos\theta\cos\phi \, , \quad x^3 = \cos\theta\sin\phi \ . \tag{2.3}$$

Then the action (2.1) takes the form

$$S = 2g \int d\tilde{\sigma}^+ d\tilde{\sigma}^- (-\partial_{\tilde{\sigma}^+} t \partial_{\tilde{\sigma}^-} t + \cos^2\theta \partial_{\tilde{\sigma}^+}\phi\partial_{\tilde{\sigma}^-}\phi + \partial_{\tilde{\sigma}^+}\theta\partial_{\tilde{\sigma}^-}\theta) \, , \tag{2.4}$$

where we have introduced the light-cone coordinates[5]

$$\tilde{\sigma}^\pm = \tilde{\sigma}^0 \pm \tilde{\sigma}^1 \ . \tag{2.5}$$

Further, since the action (2.4) is defined with fixed form of the world-sheet metric the theory has to be accompanied with the corresponding Virasoro constrains

$$\begin{aligned} T_{++} &= g[-\partial_{\tilde{\sigma}^+} t\partial_{\tilde{\sigma}^+} t + \cos^2\theta\partial_{\tilde{\sigma}^+}\phi\partial_{\tilde{\sigma}^+}\phi + \partial_{\tilde{\sigma}^+}\theta\partial_{\tilde{\sigma}^+}\theta] = 0 \, , \\ T_{--} &= g[-\partial_{\tilde{\sigma}^-} t\partial_{\tilde{\sigma}^-} t + \cos^2\theta\partial_{\tilde{\sigma}^-}\phi\partial_{\tilde{\sigma}^-}\phi + \partial_{\tilde{\sigma}^-}\theta\partial_{\tilde{\sigma}^-}\theta] = 0 \ . \end{aligned} \tag{2.6}$$

Our goal is to consider state with a constant spin density $J_{12} = J$. Let us start with solution: $\frac{d\phi}{d\tilde{\sigma}^0} = 1$ and $\theta = 0$ and perform a boost on the world-sheet coordinates $\tilde{\sigma}^\pm = \tilde{\sigma}^0 \pm \tilde{\sigma}^1$ and expand in small fluctuations around the constant spin-density solution

$$\begin{aligned} \tilde{\sigma}^+ &= 2\sqrt{g}\sigma^+ \, , \quad \tilde{\sigma}^- = \frac{\sigma^-}{2\sqrt{g}} \, , \\ \phi &= \tilde{\sigma}^0 + \frac{\delta}{\sqrt{g}} = \frac{\tilde{\sigma}^+ + \tilde{\sigma}^-}{2} + \frac{\delta}{\sqrt{g}} = \sqrt{g}\sigma^+ + \frac{\chi}{\sqrt{g}} \, , \\ \chi &= \frac{\sigma^-}{4} + \delta \, , \\ \theta &= \frac{y}{\sqrt{g}} \, , \\ t &= \tilde{\sigma}^0 = \frac{1}{2}(\tilde{\sigma}^+ + \tilde{\sigma}^-) = \sigma^+ + \frac{\sigma^-}{4\sqrt{g}} \, , \quad g \to \infty \, , \end{aligned} \tag{2.7}$$

where $\sigma^\pm$ are the light-cone coordinates after performing the boost. Note that we are interested in solutions where $\chi = \frac{1}{4}\sigma^- + \delta$ with δ representing small fluctuations. Then the rescaling (2.7) implies

$$\begin{aligned} d\tilde{\sigma}^+ d\tilde{\sigma}^- &= d\sigma^+ d\sigma^- \, , \\ \frac{\partial}{\partial\tilde{\sigma}^+} &= \frac{1}{2\sqrt{g}}\frac{\partial}{\partial\sigma^+} \equiv \frac{1}{2\sqrt{g}}\partial_+ \, , \\ \frac{\partial}{\partial\tilde{\sigma}^-} &= 2\sqrt{g}\frac{\partial}{\partial\sigma^-} \equiv 2\sqrt{g}\partial_- \ . \end{aligned} \tag{2.8}$$

[5] In the light-cone frame the metric components are $\eta^{+-} = \eta^{-+} = -2$, with the inverse $\eta_{+-} = \eta_{-+} = -\frac{1}{2}$ and with the corresponding determinant $\sqrt{-\eta} = \frac{1}{2}$.

Using (2.7) and (2.8) the action (2.4), up to constant and total derivative terms, is finite

$$S = 2 \int d\sigma^+ d\sigma^- [\partial_+ \chi \partial_- \chi + \partial_+ y \partial_- y - y^2 \partial_- \chi] \ . \tag{2.9}$$

Finally, to generalize to the case of $O(N)$ sigma model we replace y^2 with $y^2 \to \vec{y}^2 \equiv y^i y_i \, , i = 1, \dots, N-2$ in the above action and we obtain

$$S = 2 \int d\sigma^+ d\sigma^- [\partial_- \chi \partial_+ \chi + \partial_+ \vec{y} \partial_- \vec{y} - \vec{y}^2 \partial_- \chi] \ . \tag{2.10}$$

Note also that under the rescaling (2.7), T_{++} given in (2.6) takes the form

$$\lim_{g \to \infty} T_{++} = \frac{1}{2}\left(\partial_+ \chi - \frac{\vec{y}^2}{2}\right) + O\left(\frac{1}{g}\right) \equiv \frac{1}{2} j_+ + O\left(\frac{1}{g}\right) \tag{2.11}$$

and hence the Virasoro constraint $T_{++} = 0$ implies the constraint

$$j_+ \equiv \partial_+ \chi - \frac{\vec{y}^2}{2} = 0 \ . \tag{2.12}$$

In the same way the rescaling (2.7) performed on T_{--} implies

$$\partial_- \chi \partial_- \chi + \partial_- \vec{y} \partial_- \vec{y} = 16 \ . \tag{2.13}$$

In summary, we have two constraints in the theory

$$\begin{aligned} \Phi_1 &= \partial_- \chi \partial_- \chi + \partial_- \vec{y} \partial_- \vec{y} - 16 = 0 \, , \\ \Phi_2 &= \partial_+ \chi - \frac{\vec{y}^2}{2} = 0 \ . \end{aligned} \tag{2.14}$$

Following [12] we can move to a gauge fixed Lagrangian by defining new coordinates

$$x^+ \equiv \sigma^+ \, , \quad x^- \equiv \frac{\sigma^-}{2} + 2\chi \ . \tag{2.15}$$

Then it is easy to see that

$$\begin{aligned} \partial_+ &= \partial_{x^+} + 2 \partial_+ \chi \partial_{x^-} = \partial_{x^+} + \vec{y}^2 \partial_{x^-} \, , \\ \partial_- &= 2\left(\frac{1}{4} + \partial_- \chi\right) \partial_{x^-} \, , \end{aligned} \tag{2.16}$$

where we have used (2.14). Note also that Φ_2 implies

$$\partial_- \chi = \frac{\frac{1}{4} - (\partial_{x^-} \vec{y})^2}{1 + 4 (\partial_{x^-} \vec{y})^2} \, , \quad \frac{\partial_- \chi}{\frac{1}{4} + \partial_- \chi} = 2\left(\frac{1}{4} - (\partial_{x^-} \vec{y})^2\right) \ . \tag{2.17}$$

Let us now consider the equations of motion for y^i that follow from the action (2.10)

$$\partial_+ \partial_- y^i + y^i \partial_- \chi = 0 \ . \tag{2.18}$$

Then using (2.16) and (2.17) we can map (2.18) into

$$\partial_{x^-}\partial_{x^+}y^i + \partial_{x^-}(\vec{y}^2\partial_{x^-}y^i) + y^i\left(\frac{1}{4} - (\partial_{x^-}y)^2\right) = 0 \ . \qquad (2.19)$$

Then it is easy to see that these equations of motion follow from the variation of the action

$$S = 2\int dx^+dx^-\left[\partial_{x^+}\vec{y}\partial_{x^-}\vec{y} - \frac{1}{4}\vec{y}^2 + \vec{y}^2(\partial_{x^-}\vec{y}\partial_{x^-}\vec{y})\right] \ . \qquad (2.20)$$

The action (2.20) will be starting point for the Hamiltonian treatment of the reduced model. Before we proceed to this question we introduce the Lax connection for the theory given above.

3. Lax connection

In this section we introduce Lax connection that was given in [12]. As was argued there Lax connection can be obtained by taking a simple limit of the connection of $O(N)$ theory. In order to write it explicitly, we select one of the $O(N)$ generators J_{12} and consider the off-diagonal generators that mix $(1,2)$ plane with the rest. We denote these generators as $J^{\pm i}$, $i = 1,\dots,N-2$. We also need following commutation relations:

$$\begin{aligned}&\left[J^{12}, J^{\pm i}\right] = \pm J^{\pm i} \ , \\ &\left[J^{+i}, J^{-j}\right] = \delta_{ij}J^{12} - J^{ij} \ , \quad \left[J^{-i}, J^{+j}\right] = -\delta_{ij}J^{12} - J^{ij} \ . \end{aligned} \qquad (3.1)$$

Flat connection introduced in [12] is derived by taking the limit of the connection for $O(N)$ theory [21] and it takes the form

$$\begin{aligned}\mathcal{A}_+ &= \frac{i}{\sqrt{2}}\left[e^{-i\sigma^+w}y^iJ^{+i} + e^{i\sigma^+w}y^iJ^{-i}\right] \ , \\ \mathcal{A}_- &= \frac{1}{w}\left[-i\partial_-\chi J^{12} - \frac{1}{\sqrt{2}}e^{-i\sigma^+w}\partial_-y^iJ^{+i} + \frac{1}{\sqrt{2}}e^{i\sigma^+w}\partial_-y^iJ^{-i}\right] \ , \end{aligned} \qquad (3.2)$$

where w is spectral parameter. Then using the constraint $j_+ = 0$ and the equations of motion for y^i (2.18) together with (3.1) we obtain

$$\partial_+\mathcal{A}_- - \partial_-\mathcal{A}_+ + [\mathcal{A}_+, \mathcal{A}_-] = 0 \ . \qquad (3.3)$$

In other words $\mathcal{A}$ given (3.2) defines flat Lax connection. As the next step we use (2.15) and write

$$\begin{aligned}dx^+ &= d\sigma^+ \ , \\ dx^- - \vec{y}^2dx^+ &= d\sigma^- 2\left(\frac{1}{4} + \partial_-\chi\right) \ . \end{aligned} \qquad (3.4)$$

Then we obtain

$$\mathcal{A} = \mathcal{A}_+d\sigma^+ + \mathcal{A}_-d\sigma^- = \mathcal{A}_+dx^+ + \tilde{\mathcal{A}}(dx^- - \vec{y}^2dx^+) \ , \qquad (3.5)$$

where we have defined

$$\tilde{\mathcal{A}} \equiv \mathcal{A}_- \frac{1}{2(\frac{1}{4}+\partial_-\chi)} =$$
$$= \frac{1}{w}\left[-i\left(\frac{1}{4}-(\partial_{x^-}\vec{y})^2\right)J^{12} - \frac{1}{\sqrt{2}}e^{-ix^+w}\partial_{x^-}y^iJ^{+i} + \frac{1}{\sqrt{2}}e^{ix^+w}\partial_{x^-}y^iJ^{-i}\right], \quad (3.6)$$

where in the final step we used (2.16) and (2.17). Finally, we can perform gauge transformation to remove the constant part of the connection

$$\mathcal{A}' = g^{-1}\mathcal{A}g + g^{-1}dg\,, \quad g = e^{i\frac{1}{4w}x^-J^{12}} \quad (3.7)$$

and we obtain

$$\mathcal{A}'_+ = \frac{i}{\sqrt{2}}\left[e^{-ix^+w-\frac{i}{4w}x^-}y^iJ^{+i} + e^{ix^+w+i\frac{1}{4w}x^-}y^iJ^{-i}\right],$$
$$\tilde{\mathcal{A}}' = \frac{1}{w}\left[i(\partial_{x^-}y)^2J^{12} - \frac{1}{\sqrt{2}}e^{-ix^+w-i\frac{1}{4w}x^-}\partial_{x^-}y^iJ^{+i} + + \frac{1}{\sqrt{2}}e^{ix^+w+i\frac{1}{4w}x^-}\partial_{x^-}y^iJ^{-i}\right]. \quad (3.8)$$

In what follows we use the Lax connection (3.8) where we will write $\mathcal{A}$ instead of $\mathcal{A}'$. Let us now rewrite the Lax connection as

$$\mathcal{A} = (\mathcal{A}_+ - \vec{y}^2\tilde{\mathcal{A}}_-)dx^+ + \tilde{\mathcal{A}}_-dx^- =$$
$$= (\mathcal{A}_+ - \vec{y}^2\tilde{\mathcal{A}}_- + \tilde{\mathcal{A}}_-)d\tau + (\mathcal{A}_+ - \vec{y}^2\tilde{\mathcal{A}}_- - \tilde{\mathcal{A}}_-)d\sigma\,, \quad (3.9)$$

where we have introduced σ, τ defined as

$$x^\pm = \tau \pm \sigma\ . \quad (3.10)$$

We see that the spatial component of Lax connection $\mathcal{A}_\sigma$ takes the form

$$\mathcal{A}_\sigma = (\mathcal{A}_+ - \vec{y}^2\tilde{\mathcal{A}}_- - \tilde{\mathcal{A}}_-) =$$
$$= \frac{i}{\sqrt{2}}\left[e^{-ix^+w-\frac{i}{4w}x^-}y^iJ^{+i} + e^{ix^+w+i\frac{1}{4w}x^-}y^iJ^{-i}\right] - \quad (3.11)$$
$$-(1+\vec{y}^2)\frac{1}{w}\left[i(\partial_{x^-}y)^2J^{12} - \frac{1}{\sqrt{2}}e^{-ix^+w-i\frac{1}{4w}x^-}\partial_{x^-}y^iJ^{+i} + \frac{1}{\sqrt{2}}e^{ix^+w+i\frac{1}{4w}x^-}\partial_{x^-}y^iJ^{-i}\right].$$

The spatial component of Lax connection given above will be the central object for the study of the integrability of the theory. Explicitly, we will calculate the Poisson bracket between these components for different spectral parameters w, v. Before we proceed to this calculation we have to develop corresponding Hamiltonian formalism.

4. Hamiltonian formalism

Our goal is to develop the Hamiltonian formalism for the action

$$S = 4\int dx^+dx^-\sqrt{-\eta}[\partial_{x^+}\vec{y}\partial_{x^-}\vec{y} - \frac{1}{4}\vec{y}^2 + \vec{y}^2(\partial_{x^-}\vec{y}\partial_{x^-}\vec{y})]\ . \quad (4.1)$$

If we again introduce coordinates τ, σ as

$$x^{\pm} = \tau \pm \sigma \tag{4.2}$$

and consequently

$$\partial_{x^+} = \frac{1}{2}(\partial_\tau + \partial_\sigma) \, , \quad \partial_{x^-} = \frac{1}{2}(\partial_\tau - \partial_\sigma) \tag{4.3}$$

we obtain

$$S = \int d\tau d\sigma [(\partial_\tau \vec{y})^2 - (\partial_\sigma \vec{y})^2 - \vec{y}^2 + \vec{y}^2(\partial_\tau \vec{y} - \partial_\sigma \vec{y})^2] \ . \tag{4.4}$$

Then the momentum π_i conjugate to y^i takes the form

$$\pi_i = \frac{\delta L}{\delta \partial_\tau y^i} = 2[\partial_\tau y^i + \vec{y}^2(\partial_\tau y^i - \partial_\sigma y^i)] \tag{4.5}$$

and we have following canonical Poisson brackets

$$\left\{ y^i(\sigma), \pi_j(\sigma') \right\} = \delta^i_j \delta(\sigma - \sigma') \ . \tag{4.6}$$

Using (4.5) we can express $\partial_\tau y^i$ as function of π_i and $\partial_\sigma y^i$

$$\partial_\tau y^i = \frac{\frac{1}{2}\pi_i + \vec{y}^2 \partial_\sigma y^i}{1 + \vec{y}^2} \ . \tag{4.7}$$

Then corresponding Hamiltonian density takes the form

$$\mathcal{H} = \partial_\tau y^i \pi_i - \mathcal{L} = \frac{\vec{\pi}^2}{4(1+\vec{y}^2)} + \frac{\vec{y}^2(\partial_\sigma \vec{y} \vec{\pi})}{1+\vec{y}^2} - \frac{\vec{y}^2(\partial_\sigma \vec{y})^2}{1+\vec{y}^2} + (\partial_\sigma \vec{y})^2 + \vec{y}^2. \tag{4.8}$$

Finally, we use the relation (4.7) to express (3.11) as a function of canonical variables

$$\begin{aligned}
\mathcal{A}_\sigma =& \frac{i}{\sqrt{2}} \left[e^{-ix^+ w - \frac{i}{4w} x^-} y^i J^{+i} + e^{ix^+ w + i\frac{1}{4w} x^-} y^i J^{-i} \right] - \\
& - \frac{1}{w} \left[\frac{i}{4(1+\vec{y}^2)} \left(\frac{1}{2}\vec{\pi} - \partial_\sigma \vec{y} \right)^2 J^{12} - \frac{1}{2\sqrt{2}} e^{-ix^+ w - i\frac{1}{4w} x^-} \left(\frac{1}{2}\pi_i - \partial_\sigma y^i \right) J^{+i} + \right. \\
& \left. + \frac{1}{2\sqrt{2}} e^{ix^+ w + i\frac{1}{4w} x^-} \left(\frac{1}{2}\pi_i - \partial_\sigma y^i \right) J^{-i} \right] \ .
\end{aligned} \tag{4.9}$$

Now we are ready to calculate the Poisson brackets

$$\left\{ \mathcal{A}_{\sigma,\alpha\beta}(\sigma, w), \mathcal{A}_{\sigma,\gamma\delta}(\sigma', v) \right\} \ , \tag{4.10}$$

where α, β and γ, δ label matrix indices of generators J's. Since in the following we will consider the spatial components of Lax connection only we omit the subscript σ. Explicitly we define $\mathcal{A}_{\sigma,\alpha\beta} \equiv \mathcal{A}_{\alpha\beta}$.

JHEP09(2007)100

Now using the canonical Poisson brackets determined above we calculate the Poisson bracket of spatial component of Lax connection. Using

$$\begin{aligned}
&\left\{\left(\frac{1}{2}\vec{\pi}-\partial_\sigma\vec{y}\right)^2(\sigma),\left(\frac{1}{2}\pi_i-\partial_{\sigma'}y^i\right)(\sigma')\right\} = \left(\frac{1}{2}\pi-\partial_\sigma y^i\right)(\sigma)[\partial_{\sigma'}\delta(\sigma-\sigma')-\partial_\sigma\delta(\sigma-\sigma')]\,,\\
&\left\{\left(\frac{1}{2}\pi_i-\partial_\sigma y^i\right)(\sigma),\left(\frac{1}{2}\vec{\pi}-\partial_{\sigma'}\vec{y}\right)^2(\sigma')\right\} = -\left(\frac{1}{2}\pi_i-\partial_{\sigma'}y^i\right)(\sigma')[\partial_\sigma\delta(\sigma-\sigma')-\partial_{\sigma'}\delta(\sigma-\sigma')]\,,\\
&\left\{\left(\frac{1}{2}\vec{\pi}-\partial_\sigma\vec{y}\right)^2(\sigma),\left(\frac{1}{2}\vec{\pi}-\partial_{\sigma'}\vec{y}\right)^2(\sigma')\right\} = 2\left(\frac{1}{2}\pi_i-\partial_\sigma y^i\right)(\sigma)[\partial_{\sigma'}\delta(\sigma-\sigma')-\\
&\qquad\qquad -\partial_\sigma\delta(\sigma-\sigma')]\left(\frac{1}{2}\pi_i-\partial_{\sigma'}y^i\right)(\sigma') \qquad (4.11)
\end{aligned}$$

we obtain, after straightforward calculations, following result

$$\begin{aligned}
\left\{\mathcal{A}_{\alpha\beta}(\sigma,w),\mathcal{A}_{\gamma\delta}(\sigma',v)\right\} &= \mathbf{A}_{\alpha\gamma,\beta\delta}(\sigma,w,v)\delta(\sigma-\sigma')+\mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma,\sigma',w,v)\partial_{\sigma'}\delta(\sigma-\sigma')+\\
&+\mathbf{C}_{\alpha\gamma,\beta\delta}(\sigma,\sigma',w,v)\partial_\sigma\delta(\sigma-\sigma')\,, \qquad (4.12)
\end{aligned}$$

where

$$\begin{aligned}
&\mathbf{A}_{\alpha\gamma,\beta\delta}(\sigma,w,v) =\\
&= \frac{i(w-v)}{8vw}e^{-ix^+(v+w)-\frac{i(v+w)}{4vw}x^-}J^{+i}_{\alpha\beta}J^{+i}_{\gamma\delta}-\frac{i(v+w)}{8vw}e^{-ix^+(w-v)-\frac{i(v-w)}{4vw}x^-}J^{+i}_{\alpha\beta}J^{-i}_{\gamma\delta}+\\
&+\frac{i(v+w)}{8vw}e^{i(w-v)x^++\frac{i(v-w)}{4wv}x^-}J^{-i}_{\alpha\beta}J^{+i}_{\gamma\delta}-\frac{i(w-v)}{8wv}e^{ix^+(v+w)+\frac{i(v+w)}{4vw}x^-}J^{-i}_{\alpha\beta}J^{-i}_{\gamma\delta}+\\
&+\frac{i}{4w\sqrt{2}(1+\vec{y}^2)}\left[\left(\frac{1}{2}\pi_i-\partial_\sigma y^i\right)e^{-ix^+v-\frac{i}{4v}x^-}J^{12}_{\alpha\beta}J^{+i}_{\gamma\delta}+e^{ix^+v+\frac{i}{4v}x^-}\left(\frac{1}{2}\pi_i-\partial_\sigma y^i\right)J^{12}_{\alpha\beta}J^{-i}_{\gamma\delta}\right]-\\
&-\frac{i}{4v\sqrt{2}(1+\vec{y}^2)}\left[\left(\frac{1}{2}\pi_i-\partial_\sigma y^i\right)e^{-ix^+w-\frac{i}{4w}x^-}J^{+i}_{\alpha\beta}J^{12}_{\gamma\delta}+e^{ix^+w+\frac{i}{4w}x^-}\left(\frac{1}{2}\pi_i-\partial_\sigma y^i\right)J^{-i}_{\alpha\beta}J^{12}_{\gamma\delta}\right]+\\
&+\frac{i}{16vw}\frac{1}{(1+\vec{y}^2)^2}\left(\frac{1}{2}\vec{\pi}-\partial_\sigma\vec{y}\right)^2\left[e^{-ix^+v-\frac{i}{4v}x^-}y^iJ^{12}_{\alpha\beta}J^{+i}_{\gamma\delta}-e^{ix^+v+\frac{i}{4v}x^-}y^iJ^{12}_{\alpha\beta}J^{-i}_{\gamma\delta}-\right.\\
&\left.-e^{-ix^+w-\frac{i}{4w}x^-}y^iJ^{+i}_{\alpha\beta}J^{12}_{\gamma\delta}+e^{ix^+w+\frac{i}{4w}x^-}y^iJ^{-i}_{\alpha\beta}J^{12}_{\gamma\delta}\right]\,, \qquad (4.13)
\end{aligned}$$

$$\begin{aligned}
&\mathbf{C}_{\alpha\gamma,\beta\delta}(\sigma,\sigma',w,v) =\\
&= -\frac{1}{16vw}[e^{-ix^+w-\frac{i}{4w}x^-}J^{+i}_{\alpha\beta}-e^{ix^+w+\frac{i}{4w}x^-}J^{-i}_{\alpha\beta}](\sigma)\times\\
&\times[e^{-ix^+v-\frac{i}{4v}x^-}J^{+i}_{\gamma\delta}-e^{ix^+v+\frac{i}{4v}x^-}J^{-i}_{\gamma\delta}](\sigma')+\\
&+\frac{i}{16vw}\frac{1}{(1+\vec{y}^2)}(\frac{1}{2}\pi_i-\partial_\sigma y^i)(\sigma)[e^{-ix^+v-\frac{i}{4v}x^-}J^{12}_{\alpha\beta}J^{+i}_{\gamma\delta}-e^{ix^+v+\frac{i}{4v}x^-}J^{12}_{\alpha\beta}J^{-i}_{\gamma\delta}](\sigma')+\\
&+\frac{i}{16vw}[e^{-ix^+w-\frac{i}{4w}x^-}J^{+i}_{\alpha\beta}J^{12}_{\gamma\delta}-e^{ix^+w+\frac{i}{4w}x^-}J^{-i}_{\alpha\beta}J^{12}_{\gamma\delta}](\sigma)\frac{(\frac{1}{2}\pi_i-\partial_{\sigma'}y^i)}{(1+\vec{y}^2)}(\sigma')-\\
&-\frac{1}{8wv}\frac{(\frac{1}{2}\pi_i-\partial_\sigma y^i)}{1+\vec{y}^2}(\sigma)\frac{(\frac{1}{2}\pi_i-\partial_{\sigma'}y^i)}{1+\vec{y}^2}(\sigma')J^{12}_{\alpha\beta}J^{12}_{\gamma\delta} \qquad (4.14)
\end{aligned}$$

and

$$
\begin{aligned}
&\mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma,\sigma',w,v) = \\
&= \frac{1}{16vw}[e^{-ix^+w-\frac{i}{4w}x^-}J^{+i}_{\alpha\beta} - e^{ix^+w+\frac{i}{4w}x^-}J^{-i}_{\alpha\beta}](\sigma)\times \\
&\quad \times[e^{-ix^+v-\frac{i}{4v}x^-}J^{+i}_{\gamma\delta} - e^{ix^+v+\frac{i}{4v}x^-}J^{-i}_{\gamma\delta}](\sigma') - \\
&\quad -\frac{i}{16vw}\frac{1}{(1+\vec{y}^2)}(\frac{1}{2}\pi_i - \partial_\sigma y^i)(\sigma)[e^{-ix^+v-\frac{i}{4v}x^-}J^{12}_{\alpha\beta}J^{+i}_{\gamma\delta} - e^{ix^+v+\frac{i}{4v}x^-}J^{12}_{\alpha\beta}J^{-i}_{\gamma\delta}](\sigma') - \\
&\quad -\frac{i}{16vw}[e^{-ix^+w-\frac{i}{4w}x^-}J^{+i}_{\alpha\beta}J^{12}_{\gamma\delta} - e^{ix^+w+\frac{i}{4w}x^-}J^{-i}_{\alpha\beta}J^{12}_{\gamma\delta}](\sigma)\frac{(\frac{1}{2}\pi_i - \partial_{\sigma'}y^i)}{(1+\vec{y}^2)}(\sigma') + \\
&\quad +\frac{1}{8wv}\frac{(\frac{1}{2}\pi_i - \partial_\sigma y^i)}{1+\vec{y}^2}(\sigma)\frac{(\frac{1}{2}\pi_i - \partial_{\sigma'}y^i)}{1+\vec{y}^2}(\sigma')J^{12}_{\alpha\beta}J^{12}_{\gamma\delta}\ .
\end{aligned} \tag{4.15}
$$

Then it is easy to see that matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ obey the relations

$$
\mathbf{A}_{\alpha\gamma,\beta\delta}(\sigma,w,v) = -\mathbf{A}_{\gamma\alpha,\delta\beta}(\sigma,v,w) \tag{4.16}
$$

and

$$
\mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma,\sigma',w,v) = -\mathbf{C}_{\gamma\alpha,\beta\delta}(\sigma',\sigma,v,w) \tag{4.17}
$$

that are in agreement with general definition given in (A.11).

Now we are ready to exhibit the general structure of the Poisson brackets, following [22]. Let us introduce the matrices $r_{\alpha\gamma,\beta\delta}(\sigma,w,v), s_{\alpha\gamma,\beta\delta}(\sigma,w,v)$ whose explicit form in terms of matrices $\mathbf{B}_{\alpha\gamma,\beta\delta}, \mathbf{C}_{\alpha\gamma,\beta\delta}$ is given in (A.12). Then, using also the formula

$$
f(x,y)\partial_x\delta(x-y) = f(x,x)\partial_x\delta(x-y) + \partial_y f(x,y)_{y=x}\delta(x-y) \tag{4.18}
$$

we can rewrite the Poisson bracket (4.12) into the form

$$
\begin{aligned}
&\{\mathcal{A}_{\alpha\beta}(\sigma,w),\mathcal{A}_{\gamma\delta}(\sigma',v)\} = \mathbf{A}_{\alpha\gamma,\beta\delta}(\sigma,w,v)\delta(\sigma-\sigma') - \\
&\quad -\partial_u\mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma,u,w,v)_{u=\sigma}\delta(\sigma-\sigma') - \partial_u\mathbf{C}_{\alpha\gamma,\beta\delta}(u,\sigma,w,v)_{u=\sigma}\delta(\sigma-\sigma') - \\
&\quad -\mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma,\sigma,w,v)\partial_\sigma\delta(\sigma-\sigma') - \mathbf{C}_{\alpha\gamma,\beta\delta}(\sigma',\sigma',w,v))\partial_{\sigma'}\delta(\sigma-\sigma') = \\
&= (\partial_\sigma r_{\alpha\gamma,\beta\delta}(\sigma,w,v) - \partial_\sigma s_{\alpha\gamma,\beta\delta}(\sigma,w,v))\delta(\sigma-\sigma') - 2s_{\alpha\gamma,\beta\delta}(\sigma,w,v)\partial_\sigma\delta(\sigma-\sigma') + \\
&\quad +[(r_{\alpha\gamma,\sigma\delta}(\sigma,w,v) - s_{\alpha\gamma,\sigma\delta}(\sigma,w,v))\mathcal{A}_{\sigma\beta}(\sigma,w) - \\
&\quad -\mathcal{A}_{\alpha\sigma}(\sigma,w)(r_{\sigma\gamma,\beta\delta}(\sigma,w,v) - s_{\sigma\gamma,\beta\delta}(\sigma,w,v))]\delta(\sigma-\sigma') + \\
&\quad +[(r_{\alpha\gamma,\beta\sigma}(\sigma,w,v) + s_{\alpha\gamma,\beta\sigma}(\sigma,w,v))\mathcal{A}_{\sigma\delta}(v,\sigma) - \\
&\quad -\mathcal{A}_{\gamma\sigma}(v,\sigma)(r_{\alpha\sigma,\beta\delta}(\sigma,w,v) + s_{\alpha\sigma,\beta\delta}(\sigma,w,v))]\delta(\sigma-\sigma')\ .
\end{aligned} \tag{4.19}
$$

The fact that the Poisson bracket of Lax connection takes the form given above has an important consequence for the integrability of the theory. As was shown in [22] and reviewed in appendix integrable theories with the Poisson brackets of the Lax connection given in (4.19) or in its alternative form given in (A.15) possesses infinite number of conserved charges that are in involution with respect to given Poisson bracket structure. In other words we have shown that the reduced sigma model is classically integrable.

A. Review of basic properties of monodromy matrix

In this section we give a review of properties of monodromy matrix, following [23]. As opposite to this paper we will write all expressions with explicit matrix notation.

The monodromy matrix $\mathcal{T}_{\alpha\beta}(\sigma_1,\sigma_2,w)$, where w is a spectral parameter, can be defined as

$$\begin{aligned}\partial_{\sigma_1}\mathcal{T}_{\alpha\beta}(\sigma_1,\sigma_2,w) &= \mathcal{A}_{\alpha\gamma}(\sigma_1,w)\mathcal{T}_{\gamma\beta}(\sigma_1,\sigma_2,w)\,, \\ \partial_{\sigma_2}\mathcal{T}_{\alpha\beta}(\sigma_1,\sigma_2,w) &= -\mathcal{T}_{\alpha\gamma}(\sigma_1,\sigma_2,w)\mathcal{A}_{\gamma\beta}(\sigma_2,w)\end{aligned} \tag{A.1}$$

with the normalisation condition

$$\mathcal{T}_{\alpha\beta}(\sigma_1,\sigma_2,w) = \delta_{\alpha\beta} \tag{A.2}$$

and

$$\mathcal{T}^{-1}_{\alpha\beta}(\sigma_1,\sigma_2,w) = \mathcal{T}_{\alpha\beta}(\sigma_2,\sigma_1,w)\ . \tag{A.3}$$

Note that in our notation $\mathcal{A}_{\alpha\beta}(\sigma,w)$ is a spatial component of Lax connection.

Our goal is to calculate the Poisson bracket between $\mathcal{T}(w)$ and $\mathcal{T}(v)$. Following [23] we consider the Poisson bracket between any dynamical quantity $X_{\gamma\delta}$ and $\mathcal{T}_{\alpha\beta}(\sigma_1,\sigma_2,w)$ where $X_{\gamma\delta}$ does not depend σ_1 and σ_2

$$\{\mathcal{T}_{\alpha\beta}(\sigma_1,\sigma_2,w), X_{\gamma\delta}\} = W_{\alpha\gamma,\beta\delta}(\sigma_1,\sigma_2,w)\,, \tag{A.4}$$

where

$$\partial_{\sigma_1}X_{\gamma\delta} = 0\,, \quad \partial_{\sigma_2}X_{\gamma\delta} = 0\ . \tag{A.5}$$

If we derive (A.4) with respect σ_1 and σ_2 and use (A.1) we obtain two differential equations for $W(\sigma_1,\sigma_2)$

$$\partial_{\sigma_1}W_{\alpha\gamma,\beta\delta}(\sigma_1,\sigma_2,w) = \mathcal{A}_{\alpha\sigma}(\sigma_1,w)W_{\sigma\gamma,\beta\delta}(\sigma_1,\sigma_2,w) + \{\mathcal{A}_{\alpha\sigma}(\sigma_1,w), X_{\gamma\delta}\}\,\mathcal{T}_{\sigma\beta}(\sigma_1,\sigma_2,w) \tag{A.6}$$

and

$$\partial_{\sigma_2}W_{\alpha\gamma,\beta\delta}(\sigma_1,\sigma_2,w) = -\mathcal{T}_{\alpha\sigma}(\sigma_1,\sigma_2,w)\,\{\mathcal{A}_{\sigma\beta}(\sigma_2,w), X_{\gamma\delta}\} - W_{\alpha\gamma,\sigma\delta}(\sigma_1,\sigma_2,w)\mathcal{A}_{\sigma\beta}(\sigma_2,w). \tag{A.7}$$

The equations (A.6) and (A.7) have solution in the form

$$W_{\alpha\gamma,\beta\delta}(\sigma_1,\sigma_2,w) = \int_{\sigma_2}^{\sigma_1} d\sigma'\mathcal{T}_{\alpha\sigma_1}(\sigma_1,\sigma',w)\left\{\mathcal{A}_{\sigma_1\sigma_2}(\sigma',w), X_{\gamma\delta}\right\}\mathcal{T}_{\sigma_2\beta}(\sigma',\sigma_2,w)\ . \tag{A.8}$$

Let us now presume that $X_{\gamma\delta} = \mathcal{T}_{\gamma\delta}(\sigma'_1,\sigma'_2,v)$ where all $\sigma_1,\sigma_2,\sigma'_1$ and σ'_2 are distinct. Then (A.4) together with (A.8) implies

$$\begin{aligned}\left\{\mathcal{T}_{\alpha\beta}(\sigma_1,\sigma_2,w),\mathcal{T}_{\gamma\delta}(\sigma'_1,\sigma'_2,v)\right\} = &\int_{\sigma_2}^{\sigma_1} d\sigma\int_{\sigma'_2}^{\sigma'_1} d\sigma'\mathcal{T}_{\alpha\sigma_1}(\sigma_1,\sigma,w)\mathcal{T}_{\gamma\rho_1}(\sigma'_1,\sigma',v)\times \\ &\times\left\{\mathcal{A}_{\sigma_1\sigma_2}(\sigma,w),\mathcal{A}_{\rho_1\rho_2}(\sigma',v)\right\}\mathcal{T}_{\sigma_2\beta}(\sigma,\sigma_2,w)\mathcal{T}_{\rho_2\delta}(\sigma',\sigma'_2,v)\ .\end{aligned} \tag{A.9}$$

Let us now presume that the Poisson bracket of spatial components of Lax connection $\mathcal{A}(\sigma, w)$ and $\mathcal{A}(\sigma', v)$, where w and v are spectral parameters, takes the form

$$\begin{aligned}\left\{\mathcal{A}_{\alpha\beta}(\sigma,w),\mathcal{A}_{\gamma\delta}(\sigma',v)\right\} &= \mathbf{A}_{\alpha\gamma,\beta\delta}(\sigma,w,v)\delta(\sigma-\sigma')+\mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma,\sigma',w,v)\partial_{\sigma'}\delta(\sigma-\sigma')+\\ &+\mathbf{C}_{\alpha\gamma,\beta\delta}(\sigma,\sigma',w,v)\partial_{\sigma}\delta(\sigma-\sigma')\ .\end{aligned} \tag{A.10}$$

Then an antisymmetry of Poisson bracket implies

$$\begin{aligned}\mathbf{A}_{\alpha\gamma,\beta\delta}(\sigma,w,v) &= -\mathbf{A}_{\gamma\alpha,\delta\beta}(\sigma,v,w)\,,\\ \mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma,\sigma',w,v) &= -\mathbf{C}_{\gamma\alpha,\beta\delta}(\sigma',\sigma,v,w)\,,\\ \mathbf{C}_{\alpha\gamma,\beta\delta}(\sigma,\sigma',w,v) &= -\mathbf{B}_{\gamma\alpha,\delta\beta}(\sigma',\sigma,v,w)\ .\end{aligned} \tag{A.11}$$

Let us introduce matrices $r_{\alpha\gamma,\beta\delta}(\sigma,w,v), s_{\alpha\gamma,\beta\delta}(\sigma,w,v)$ defined as

$$\begin{aligned}s_{\alpha\gamma,\beta\delta}(\sigma,w,v) &= \frac{1}{2}[\mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma,\sigma,w,v)+\mathbf{B}_{\gamma\alpha,\delta\beta}(\sigma,\sigma,w,v)] =\\ &= \frac{1}{2}[\mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma,\sigma,w,v)-\mathbf{C}_{\alpha\gamma,\beta\delta}(\sigma,\sigma,w,v)]\,,\\ r_{\alpha\gamma,\beta\delta}(\sigma,w,v) &= \frac{1}{2}[\mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma,\sigma,w,v)-\mathbf{B}_{\gamma\alpha,\delta\beta}(\sigma,\sigma,v,w)]+\hat{r}_{\alpha\gamma,\beta\delta}(\sigma,w,v) =\\ &= \frac{1}{2}[\mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma,\sigma,w,v)+\mathbf{C}_{\alpha\gamma,\beta\delta}(\sigma,\sigma,v,w)]+\hat{r}_{\alpha\gamma,\beta\delta}(\sigma,w,v)\,,\end{aligned} \tag{A.12}$$

where $\hat{r}$ is solution of the inhomogeneous first order differential equation

$$\begin{aligned}&\partial_\sigma\hat{r}_{\alpha\gamma,\beta\delta}(\sigma,w,v)+[\hat{r}_{\alpha\gamma,\sigma\delta}(\sigma,w,v)\mathcal{A}_{\sigma\beta}(\sigma,w)-\mathcal{A}_{\alpha\sigma}(\sigma,w)\hat{r}_{\sigma\gamma,\beta\delta}(\sigma,w,v)]+\\ &\qquad+[\hat{r}_{\alpha\gamma,\beta\sigma}(\sigma,w,v)\mathcal{A}_{\sigma\delta}(v,\sigma)-\mathcal{A}_{\gamma\sigma}(v,\sigma)\hat{r}_{\alpha\sigma,\beta\delta}(\sigma,w,v)] = \Omega_{\alpha\gamma,\beta\delta}(\sigma,w,v)\,,\end{aligned} \tag{A.13}$$

where

$$\begin{aligned}\Omega_{\alpha\gamma,\beta\delta}(\sigma,w,v) &= \mathbf{A}_{\alpha\gamma,\beta\delta}(\sigma,w,v)-\partial_u(\mathbf{B}_{\alpha\gamma,\beta\delta}(\sigma,u,w,v)+\mathbf{C}_{\alpha\gamma,\beta\delta}(u,\sigma,w,v))_{u=\sigma}+\\ &+[\mathcal{A}_{\gamma\sigma}(\sigma,v)\mathbf{B}_{\alpha\sigma,\beta\delta}(\sigma,\sigma,w,v)-\mathbf{B}_{\alpha\gamma,\beta\sigma}(\sigma,\sigma,w,v)\mathcal{A}_{\sigma\delta}(\sigma,v)]+\\ &+[\mathcal{A}_{\alpha\sigma}(\sigma,w)\mathbf{C}_{\sigma\gamma,\beta\delta}(\sigma,\sigma,w,v)-\mathbf{C}_{\alpha\gamma,\sigma\delta}(\sigma,\sigma,w,v)\mathcal{A}_{\sigma\beta}(\sigma,w)]\ .\end{aligned} \tag{A.14}$$

Then we can rewrite the Poisson bracket (4.12) in the form

$$\begin{aligned}\left\{\mathcal{A}_{\alpha\beta}(w,\sigma),\mathcal{A}_{\gamma\delta}(v,\sigma')\right\} =& [r_{\alpha\gamma,\rho\delta}(w,v,\sigma)\mathcal{A}_{\rho\beta}(\sigma,w)-\mathcal{A}_{\alpha\rho}(\sigma,w)r_{\rho\beta,\gamma\delta}(w,v,\sigma)+\\ &+r_{\alpha\gamma,\beta\sigma}(w,v,\sigma)\mathcal{A}_{\sigma\delta}(\sigma,v)-\mathcal{A}_{\gamma\sigma}(\sigma,v)r_{\alpha\sigma,\beta\delta}(w,v,\sigma)+\\ &+s_{\alpha\gamma,\rho\delta}(w,v,\sigma)\mathcal{A}_{\rho\beta}(\sigma,w)-\mathcal{A}_{\alpha\rho}(\sigma,w)s_{\rho\beta,\gamma\delta}(w,v,\sigma)-\\ &-s_{\alpha\gamma,\beta\sigma}(w,v,\sigma)\mathcal{A}_{\sigma\delta}(\sigma,v)-\mathcal{A}_{\gamma\sigma}(\sigma,v)s_{\alpha\sigma,\beta\delta}(w,v,\sigma)]\,\delta(\sigma-\sigma')-\\ &-(r(\sigma,w,v)+s(\sigma,w,v)-r(\sigma',w,v)+s(\sigma',w,v))_{\alpha\gamma,\beta\delta}\partial_\sigma\delta(\sigma-\sigma').\end{aligned} \tag{A.15}$$

Let us now return to the equation (A.13). The general solution of this equation takes the form

$$\hat{r}_{\alpha\gamma,\beta\delta}(\sigma,w,v)=\int_a^\sigma d\sigma' \mathcal{T}_{\alpha\sigma_1}(\sigma,\sigma',w)\mathcal{T}_{\gamma\sigma_1}(\sigma,\sigma',v)\Omega_{\sigma_1\sigma_2,\rho_1\rho_2}(\sigma',w,v)\mathcal{T}_{\rho_1\beta}(\sigma',\sigma,w)\mathcal{T}_{\rho_2\delta}(\sigma',\sigma,v)$$
$$+\mathcal{T}_{\alpha\sigma_1}(\sigma,a,w)\mathcal{T}_{\beta\sigma_2}(\sigma,a,v)\hat{N}_{\sigma_1\sigma_2,\rho_1\rho_2}(a,w,v)\mathcal{T}_{\rho_1\beta}(a,\sigma,w)\mathcal{T}_{\rho_2\gamma}(a,\sigma,v)\,, \tag{A.16}$$

where a is arbitrary real number and $\hat{N}_{\alpha\beta,\gamma\delta}(w,v,a)$ is an arbitrary σ-independent matrix that satisfy the relation

$$\hat{N}_{\gamma\alpha,\delta\beta}(a,w,v) = -\hat{N}_{\alpha\gamma,\beta\delta}(a,v,w)\ . \tag{A.17}$$

Note that the $\hat{N}$ part of $\hat{r}$ is solution of the homogeneous equation associated with (A.13) and has to be determined by choice of boundary conditions for $\hat{r}(\sigma,w,v)$. We would like also to mention that $\hat{r}$ can be non-local expressions in terms of canonical variables of theory. Then, when it is possible, we can choose $\hat{N}$ such that $\hat{r}$ be a local matrix in terms of field of the theory.

Using of the form of the Poisson bracket (A.15) we can calculate the algebra of monodromy matrices when $\sigma_1,\sigma_2,\sigma_1',\sigma_2'$ are all different. We obtain, if σ_1 and σ_1' are larger than σ_2 and σ_2', $\sigma_1^0=\min(\sigma_1,\sigma_1')$, $\quad\sigma_2^0=\max(\sigma_2,\sigma_2')$

$$\begin{aligned}
&\left\{\mathcal{T}_{\alpha\beta}(\sigma_1,\sigma_2,w),\mathcal{T}_{\gamma\delta}(\sigma_1',\sigma_2',v)\right\} = \\
&\quad = \mathcal{T}_{\alpha\sigma_1}(\sigma_1,\sigma_1^0,w)\mathcal{T}_{\gamma\sigma_2}(\sigma_1',\sigma_1^0,v)\left[r(\sigma_1^0,w,v)+\epsilon(\sigma_1-\sigma_1')s(\sigma_1^0,w,v)\right]_{\sigma_1\sigma_2,\rho_1\rho_2}\times \\
&\qquad \times\mathcal{T}_{\rho_1\beta}(\sigma_1^0,\sigma_2,w)\mathcal{T}_{\rho_2\delta}(\sigma_1^0,\sigma_2',v) - \\
&\qquad -\mathcal{T}_{\alpha\sigma_1}(\sigma_1,\sigma_2^0,w)\mathcal{T}_{\gamma\sigma_2}(\sigma_1',\sigma_2^0,v)\left[r(\sigma_2^0,w,v)+\epsilon(\sigma_2'-\sigma_2)s(\sigma_2^0,w,v)\right]_{\sigma_1\sigma_2,\rho_1\rho_2}\times \\
&\qquad \times\mathcal{T}_{\rho_1\beta}(\sigma_2^0,\sigma_2,w)\mathcal{T}_{\rho_2\delta}(\sigma_2^0,\sigma_2',v)\,,
\end{aligned} \tag{A.18}$$

where $\epsilon(x)=\mathrm{sign}(x)$. It is important to note that in the non-ultralocal case the algebra (A.18), due to the presence of the s-term, the function

$$\triangle^{(1)}_{\alpha\gamma,\beta\delta}(\sigma_1,\sigma_2,\sigma_1',\sigma_2',w,v) = \left\{\mathcal{T}_{\alpha\beta}(\sigma_1,\sigma_2,w),\mathcal{T}_{\gamma\delta}(\sigma_1',\sigma_2',v)\right\} \tag{A.19}$$

is well defined and continuous where $\sigma_1,\sigma_2,\sigma_1',\sigma_2'$ are all distinct, but it has discontinuities proportional to $2s$ across the hyperplanes corresponding to some of the $\sigma_1,\sigma_2,\sigma_1',\sigma_2'$ being equal. Then if we want to define the Poisson bracket of transfer matrices for coinciding intervals ($\sigma_1=\sigma_1',\sigma_2=\sigma_2'$) or adjacent intervals ($\sigma_1'=\sigma_2$ or $\sigma_1=\sigma_1'$) requires the value of the discontinuous matrix-valued function $\triangle^{(1)}$ at its discontinuities. It was shown in [22] that requiring anti-symmetry of the Poisson bracket and the derivation rule to hold imposes the symmetric definition of $\triangle^{(1)}$ at its discontinuous points. For example, at $\sigma_1=\sigma_1'$ we must define

$$\triangle^{(1)}_{\alpha\gamma,\beta\delta}(\sigma_1,\sigma_2,\sigma_1,\sigma_2',w,v) = \lim_{\epsilon\to0^+}\frac{1}{2}(\triangle^{(1)}_{\alpha\gamma,\beta\delta}(\sigma_1,\sigma_2,\sigma_1+\epsilon,\sigma_2,w,v)+$$
$$+\triangle^{(1)}_{\alpha\gamma,\beta\delta}(\sigma_1,\sigma_2,\sigma_1-\epsilon,\sigma_2',w,v)) \tag{A.20}$$

and likewise for all other possible coinciding endpoints. This definition of $\triangle^{(1)}$ at its discontinuities implies an definition of the Poisson bracket between transition matrices for coinciding and adjacent intervals that is consistent with the anti-symmetry of the Poisson bracket and the derivation rule. However as was shown in [23][6] this definition of the Poisson bracket $\left\{\mathcal{T} \overset{\otimes}{,} \mathcal{T}\right\}$ [7] does not satisfy the Jacobi identity so that in fact no strong definition of the bracket $\left\{\mathcal{T} \overset{\otimes}{,} \mathcal{T}\right\}$ with coinciding or adjecent intervals can be given without violating the Jacobi identity. However, as was further shown in [23] it is possible to give a weak definition of this bracket for coinciding or adjacent intervals as well.[8] We are not going into details of the procedure, interesting reader can read the original paper [23] or more recent [24]. For example, it was shown that the algebra of two $\mathcal{T}$'s for equal intervals takes the form

$$\begin{aligned}\{\mathcal{T}_{\alpha\beta}(\sigma_1,\sigma_2,w),\mathcal{T}_{\gamma\delta}(\sigma_1,\sigma_2,v)\} &= r_{\alpha\gamma,\sigma\rho}(\sigma_1,w,v)\mathcal{T}_{\sigma\beta}(\sigma_1,\sigma_2,w)\mathcal{T}_{\rho\delta}(\sigma_1,\sigma_2,v) - \\ &\quad -\mathcal{T}_{\alpha\sigma}(\sigma_1,\sigma_2,w)\mathcal{T}_{\gamma\rho}(\sigma_1,\sigma_2,v)r_{\sigma\rho,\beta\delta}(\sigma_2,w,v)\,,\end{aligned} \tag{A.21}$$

where $\{,\}$ stands for the weak brackets defined in (A.20).

Let us now return to our model. Since the reduced sigma model is defined on the infinite line it is natural to introduce following object

$$\Omega(w) = \mathcal{T}(\infty,-\infty,w)\ . \tag{A.22}$$

Further, it is also natural to define

$$r(w,v) \equiv \lim_{\sigma\rightarrow\infty} r(w,v,\sigma) = \lim_{\sigma\rightarrow-\infty} r(w,v,\sigma)\ . \tag{A.23}$$

For example, this condition clearly holds for world-sheet fields that vanish at asymptotic infinity. Then using (A.21) we finally obtain

$$\begin{aligned}\{\mathrm{Tr}\Omega(w),\mathrm{Tr}\Omega(v)\} &= \{\Omega_{\alpha\alpha}(w),\Omega_{\beta\beta}(v)\} = \\ &= r_{\alpha\beta,\sigma_1\sigma_2}(w,v)\Omega_{\sigma_1\alpha}(w)\Omega_{\sigma_2\beta}(v) - \Omega_{\alpha\sigma_1}(w)\Omega_{\beta\sigma_2}(v)r_{\sigma_1\sigma_2,\alpha\beta}(w,v) = \\ &= r_{\alpha\beta,\sigma_1\sigma_2}(w,v)\Omega_{\sigma_1\alpha}(w)\Omega_{\sigma_2\beta}(v) - r_{\alpha\beta,\sigma_1\sigma_2}(w,v)\Omega_{\sigma_1\alpha}(w)\Omega_{\sigma_2\beta}(v) = 0\ .\end{aligned} \tag{A.24}$$

Since $\mathrm{Tr}\Omega(w)$ is generator of local conserved charges the result given above implies that these conserved charges are in involution with respect to brackets (A.20). This result implies an classical integrability of given theory.

Acknowledgments

This work was supported in part by the Czech Ministry of Education under Contract No. MSM 0021622409, by INFN, by the MIUR-COFIN contract 2003-023852 and , by the EU contracts MRTN-CT-2004-503369 and MRTN-CT-2004-512194, by the INTAS contract 03-516346 and by the NATO grant PST.CLG.978785.

[6] For very nice recent discussion, see [24].

[7] For simplicity of notation we use the double index notation $\left\{\mathcal{T}(w) \overset{\otimes}{,} \mathcal{T}(v)\right\}_{\alpha\gamma,\beta\delta} = \{\mathcal{T}_{\alpha\beta}(w),\mathcal{T}_{\gamma\delta}(v)\}$.

[8] The bracket is called to be week when the multiple bracket $\left\{\mathcal{T} \overset{\otimes}{,} \left\{\ldots\left\{\mathcal{T} \overset{\otimes}{,} \mathcal{T}\right\}\ldots\right\}\right\}$. with n factors of $\mathcal{T}$ has to be defined for each n separately.

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JHEP09(2007)100

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